Wavelets and high order numerical differentiation

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Goal

- Approximate $f^{(k)}(x)$ by using (finite) wavelet transformation
- Use Meyer wavelet function

Definition & Assumption

- $||\cdot||$: $L^2(\mathbb{R})$ norm
  $||\cdot||_p$: norm on Soblev space $H^k(\mathbb{R})$

$$||f||_p := \left(\int_{-\infty}^{\infty}(1+\xi^2)^p|\hat{f})\xi)|^2 d\xi\right)^{1/2}$$

- Assume $f \in C^k(\mathbb{R}) \bigcap H^k(\mathbb{R})$

- $f_\delta(x) \in L^2(\mathbb{R})$: measured data
- $<,>$: $L^2(\mathbb{R})$-inner product

Meyer wavelet

- $\phi(x), \psi(x)$: Meyer scaling and wavelet function
- $$\hat{\phi}(w) = \begin{cases} 1 & |x| < \frac{2}{3}\pi \\ \cos\left(\frac{\pi}{2}\nu(\frac{3}{2\pi}|w| - 1)\right) & \frac{2}{3}\pi \le |w| \le \frac{4}{3}\pi \\ 0 & |w| > \frac{4}{3}\pi \end{cases}$$

  where $\nu(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1 \end{cases}$ and $\nu(x) + \nu(1-x) = 1$ for $x \in (0,1)$

Meyer wavelet

- $\hat{\psi}, \hat{\phi}$ have compact support.
- $\nu \in C^\infty \longrightarrow \phi, \psi, \hat{\phi}, \hat{\psi} \in C^\infty$

- $V_j = \text{span}\,\{\phi_{jk} : k \in \mathbb{Z}\}$
- P_j : orthogonal projection of $g \in L^2(\mathbb{R})$ on V_j

$$P_j g := \sum_{k \in \mathbb{Z}} < g, \phi_{jk} > \phi_{jk}$$

Approximation theorem

- $f \in C^k(\mathbb{R}) \bigcap H^k(\mathbb{R})$: exact data, $f_\delta(x) \in L^2(\mathbb{R})$: noise data with

$$||f_\delta - f|| < \delta \tag{1}$$

- We want to approximate $f^{(k)}(x)$
- *A priori* condition : $f \in H^p(\mathbb{R})$ for some $p > k$ and

$$||f||_p \leq E \tag{2}$$

$D_k := \frac{d^k}{dx^k}$: differentiation operator

- We define the wavelet regularization approximation of $f^{(k)}(x) = D_k f(x)$ as

$$D_{k,j} f_\delta(x) := D_k(P_j f_\delta(x))$$

Approximation theorem

Theorem Suppose condition (2) holds for some $p > k$ and (1) holds. Take

$$J^* := \left[\log_2\left(\frac{E}{\delta}\right)^{1/p}\right] + 1$$

Then, the following inequality holds :

$$||D_k f - D_{k,J^*} f_\delta|| \leq (1 + 2C)E^{k/p}\delta^{1-k/p} \tag{3}$$

where C is a positive constant

Approximation theorem (continued)

If we choose J^{**} as

$$J^{**} := \left[\log_2 \left(\frac{1}{\delta}\right)^{1/p}\right] + 1$$

, then we get a result similar to (3)

$$||D_k f - D_{k,J^{**}} f_\delta|| \leq (C + (1 + C)E)\delta^{1-k/p} \quad (4)$$

Settings

- $f(x) : x \in [0,1]$: true function
- $0 = x_1 < \cdots < x_128 = 1$: equidistant grid.
  $F = (f(x_1), \cdots, f(x_n))$
- $F_\delta = F + \epsilon rnorm(128)$
- $\delta = \sqrt{\frac{1}{n}\sum_{l=1}^{128}(F_\delta(x_l) - F(x_l))^2}$

$$\text{result}: f(x) = \exp\left(2 - \frac{1}{x(1-x)}\right)$$